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P-stable singlestep methods for periodic initial-value problems involving second-order differential equations

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SUMMARY

A class of P-stable singlestep methods is discussed for solving initial-value problems involving second-order differential equations. The methods depend upon a parameter $p > 0$ and reduce to the classical methods for $p = 0$. A few choices of p have been discussed for which the methods are P-stable. Further, when p is chosen for linear problems as the square of the frequency of the periodic solution, the methods are P-stable. Numerical results for both linear and non-linear problems show that the P-stability is an important requirement for determining periodic numerical solutions of second-order differential equations.

1. Introduction

Second-order differential equations with periodic solutions arise in a wide variety of important physical problems. When conventional methods are applied to obtain the solution, the time increment must be limited to a value of the order of the reciprocal of the frequency of the periodic solution. Any attempt to use a larger increment results in the calculations becoming unstable and producing completely erroneous results. The Störmer-Cowell linear multistep methods of order greater than two are found to be unstable for large step sizes. Gautschi [2] and Salzer [6] have developed multistep methods based on Fourier polynomials. Stiefel and Bettis [8] have discussed the stabilisation of the Cowell method for integrating periodic initial-value problems. Hybrid multistep methods have been studied by Dyer [1]. Lambert and Watson [4] have discussed the application of symmetric multistep methods to periodic initial-value problems. Singlestep methods have received much less attention as compared with multistep methods for solving periodic initial-value problems. Scheifele [7] has developed a singlestep method based on the series expansion of the solution in terms of certain entire functions.

In this paper we develop singlestep methods of the form

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + h^2\psi_1(t_n,y_n,h), \\ y'_{n+1} &= y'_n + h\psi_2(t_n,y_n,h), \end{aligned} \tag{1.1}$$

to obtain a numerical solution of the periodic initial-value problem

$$\begin{aligned} y'' &= f(t,y), \\ y(t_0) &= y_0, y'(t_0) = y'_0. \end{aligned} \tag{1.2}$$

The functions $\psi_1(t_n,y_n,h)$ and $\psi_2(t_n,y_n,h)$ are the increment functions. If the increment functions are used with the same local truncation error of $O(h^{q+1})$ then method (1.1) has local truncation errors of $O(h^{q+1})$ in y and $O(h^q)$ in y'. The global truncation error of method (1.1) is then defined to be $O(h^q)$ and method (1.1) is said to be of order q. Let us apply method (1.1) to the initial-value problem

$$y'' = -\lambda y, \quad \lambda > 0, \quad y(t_0) = y_0, y'(t_0) = y'_0, \tag{1.3}$$

and assume that it can be written in the form

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix} = \mathbf{E}(\sqrt{\lambda}\, h) \begin{bmatrix} y_n \\ hy'_n \end{bmatrix} \tag{1.4}$$

where $\mathbf{E}(\sqrt{\lambda}\, h)$ is a 2 x 2 matrix.

Definition 1.1 Method (1.1) is said to have interval of periodicity $(0,H_0^2)$ if, for all $H^2 \in (0,H_0^2)$, $H = \sqrt{\lambda}\, h$, h being the step length, all the eigenvalues of $\mathbf{E}(\sqrt{\lambda}\, h)$ are complex and lie on the unit circle.

Definition 1.2 The singlestep method defined by (1.1) is said to be P-stable if its interval of periodicity is $(0,\infty)$.

Here, singlestep methods dependent on a parameter p are developed. The methods reduce to their classical counterparts for $p = 0$. When $p \to \infty$, methods of lower order are obtained. For p equal to the square of the frequency of the periodic solution in the linear homogeneous equation, the singlestep methods are P-stable.

2. Derivation of the methods

We write (1.2) in the form

$$y'' + py = \varphi(t,y) \tag{2.1}$$

where

$$\varphi(t,y) = f(t,y) + py \tag{2.2}$$

and $p > 0$ is an arbitrary parameter to be determined. The equation (2.1) can be written as

$$y'' + py = g(t) \tag{2.3}$$

where $g(t)$ may be considered as an approximation to the function $\varphi(t,y)$. The general solution of (2.3) will consist of a complementary function and a particular integral. We have

$$y(t) = A\cos\sqrt{p}\,t + B\sin\sqrt{p}\,t + \frac{1}{\sqrt{p}}\int_{t_n}^{t}\sin\sqrt{p}\,(t-\tau)\,g(\tau)\,d\tau \tag{2.4a}$$

where A and B are arbitrary constants. Differentiating (2.4a) with respect to t we obtain

$$y'(t) = -A\sqrt{p}\sin\sqrt{p}\,t + B\sqrt{p}\cos\sqrt{p}\,t + \int_{t_n}^{t}\cos\sqrt{p}\,(t-\tau)\,g(\tau)\,d\tau. \tag{2.4b}$$

Evaluating (2.4a) at t_{n+1}, t_n and (2.4b) at t_n and eliminating A and B from the resulting equations we obtain

$$y(t_{n+1}) = \cos\sqrt{p}h\,y(t_n) + \frac{1}{\sqrt{p}}\sin\sqrt{p}h\,y'(t_n) + \frac{1}{\sqrt{p}}\int_{t_n}^{t_{n+1}}\sin\sqrt{p}\,(t_{n+1}-\tau)\,g(\tau)\,d\tau. \tag{2.5a}$$

Evaluating (2.4a) at t_n and (2.4b) at t_{n+1}, t_n and eliminating A and B from the resulting equation we obtain

$$y'(t_{n+1}) = -\sqrt{p}\sin\sqrt{p}h\,y(t_n) + \cos\sqrt{p}h\,y'(t_n) + \int_{t_n}^{t_{n+1}}\cos\sqrt{p}\,(t_{n+1}-\tau)\,g(\tau)\,d\tau. \tag{2.5b}$$

Since $y(t)$ and $y'(t)$ are known at the initial point $t = t_n$, to derive singlestep methods for the numerical integration of equation (2.1), we replace the function $g(\tau)$ in (2.5) by an appropriate interpolating polynomial at $t = t_n$ and obtain singlestep methods of the form (1.1).

2.1 Taylor-series methods

We approximate $\varphi(t,y)$ by Taylor's interpolating polynomial of degree $q+1$ at $t = t_n$ and substitute for $g(\tau)$ in (2.5) the approximate polynomial

$$g(\tau) = \varphi(\tau,y) = \sum_{m=0}^{q}\frac{(\tau-t_n)^m}{m!}\,\varphi_n^{(m)}(t_n) + T_{q+1} \tag{2.6}$$

where

$$T_{q+1} = \frac{(\tau-t_n)^{q+1}}{(q+1)!}\,\varphi^{(q+1)}(\xi),\quad t_n < \xi < t_{n+1}. \tag{2.7}$$

Neglecting the error term and using (2.6) in (2.5) we write the singlestep methods with $\omega = \sqrt{p}h$ as

$$y_{n+1} = y_n \cos\omega + hy'_n \frac{\sin\omega}{\omega} + \sum_{m=0}^{q} h^{m+2} F_{m+2}\, \varphi_n^{(m)}, \tag{2.8a}$$

$$hy'_{n+1} = -y_n\, \omega \sin\omega + hy'_n \cos\omega + \sum_{m=0}^{q} h^{m+2} F_{m+1} \varphi_n^{(m)}, \tag{2.8b}$$

with corresponding truncation errors in y and y' being given by

$$TE_{q+1} = h^{q+3} F_{q+3}\, \varphi^{(q+1)}(\xi) + O(ph^{q+4}), \tag{2.9a}$$

$$TE'_{q+1} = h^{q+2} F_{q+2}\, \varphi^{(q+1)}(\xi) + O(ph^{q+3}), \tag{2.9b}$$

where we have used

$$F_{m+2} = \frac{1}{m!} - \frac{1}{h^{m+2}} \frac{1}{\sqrt{p}} \int_{t_n}^{t_{n+1}} \sin\sqrt{p}\,(t_{n+1} - \tau)(\tau - t_n)^m\, d\tau. \tag{2.10}$$

From (2.10) we can obtain the recurrence relation

$$\omega^2 F_{m+2} = \frac{1}{m!} - F_m, \quad m = 0, 1, 2, ..., \tag{2.11}$$

with $F_0 = \cos\omega$ and $F_1 = \dfrac{\sin\omega}{\omega}$.

It can be easily verified from (2.10) that

$$F_{m+2} \to \frac{1}{(m+2)} \text{ as } \omega \to 0, \tag{2.12a}$$

$$F_{m+2} \to 0 \text{ as } \omega \to \infty, \tag{2.12b}$$

and from (2.11)

$$\lim_{\omega\to\infty} \omega^2 F_{m+2} = \frac{1}{m!}\,. \tag{2.12c}$$

Thus the modified Taylor-series method (2.8) will become the classical Taylor-series method of order $q+1$ as $\omega \to 0$ and of order $q-1$ as $\omega \to \infty$. For arbitrary p, it is of order $q-1$.

2.2 *Runge-Kutta methods*

To avoid calculation of higher-order derivatives of $\varphi(t,y)$ in the modified Taylor-series method (2.8) we write (2.8) in the modified Runge-Kutta form (see Lawson [5]) as

$$y_{n+1} = y_n \cos\omega + hy'_n \frac{\sin\omega}{\omega} + \sum_{i=1}^{M} W_i K_i, \tag{2.13a}$$

$$hy'_{n+1} = -y_n\,\omega\sin\omega + hy'_n\cos\omega + \sum_{i=1}^{M} W_i^* K_i, \tag{2.13b}$$

where

$$L_1 = \frac{h^2}{2} f(t_n, y_n),$$

$$K_1 = L_1 + \frac{ph^2}{2} y_n,$$

$$L_i = \frac{h^2}{2} f(t_n + a_i h, Z_i),$$

$$Z_i = y_n + a_i h y'_n + \sum_{j=1}^{i-1} a_{ij} L_j,$$

$$K_i = L_i + \frac{ph^2}{2} Z_i, \quad i = 2, 3, \ldots, M.$$

The parameters a_i, W_i and $W_i^*$are determined from the equations obtained by comparing the coefficients of the various order derivatives of $\varphi(t,y)$ in (2.8) and (2.13). The abscissas $0 < a_i \leqslant 1$, $i = 2(1)M$ are specified suitably and the values W_i and W_i^*in (2.13) for $i = 1(1)$M are obtained as functions of ω. Further, we also get an implicit equation in p which for $p = 0$ is identically satisfied.

In order to get an explicit expression for the determination of p, we adopt Treanor's [9] approach and write $f(t,y)$ in (1.2) as

$$f(t,y) = -p(y - y_n) + A + B(t - t_n) + \frac{C}{2}(t - t_n)^2. \tag{2.14}$$

Evaluating (2.14) at the points t_n, $t_i = t_n + a_i h$, $i = 2,3,4$, we obtain four equations for the determination of the four unknowns p,A,B,C. Denoting $f_1 = f(t_n, y_n)$, $f_i = f(t_i, y_i)$, $\varphi_i = f_i + py_i$ we write the values for A,B,C and p as

$$A = f_1,$$

$$Bh = \{a_3^2(\varphi_2 - \varphi_1) - a_2^2(\varphi_3 - \varphi_1)\}/A_4,$$

$$\frac{Ch^2}{2} = -\{a_3(\varphi_2 - \varphi_1) - a_2(\varphi_3 - \varphi_1)\}/A_4,$$

$$p = -\left\{\frac{(f_4 - f_1)A_4 - (f_3 - f_1)A_3 + (f_2 - f_1)A_2}{(y_4 - y_1)A_4 - (y_3 - y_1)A_3 + (y_2 - y_1)A_2}\right\} \tag{2.15}$$

where

$$A_2 = a_3 a_4 (a_4 - a_3), \quad A_3 = a_2 a_4 (a_4 - a_2),$$

$$A_4 = a_2 a_3 (a_3 - a_2), \quad a_2 \neq a_3 \neq a_4.$$

If we make use of the nodes of the classical Runge-Kutta four-stage method wherein

$$a_2 = a_3 \text{ viz. } a_2 = a_3 = \tfrac{1}{2},\ \ a_{21} = \tfrac{1}{4},\ \ a_{31} = 0,\ \ a_{32} = \tfrac{1}{4}$$

$$a_4 = 1,\ \ a_{41} = a_{42} = 0,\ \ a_{43} = 1,$$

the values of A, B, C and p are obtained from (2.14) as

$$\begin{aligned} A &= f_1, \\ Bh &= -3\varphi_1 + 2\varphi_2 + 2\varphi_3 - \varphi_4, \\ Ch^2 &= 4(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4), \\ p &= -\frac{f_3 - f_2}{y_3 - y_2}\ . \end{aligned} \tag{2.16}$$

2.3 Runge-Kutta-Treanor-Methods

With the values of A,B,C and p as given by (2.15) we can write (2.14) as

$$\begin{aligned} \varphi(\tau,y) &= f(\tau,y) + py \\ &= py_n + A + B(\tau - t_n) + \frac{C}{2}(\tau - t_n)^2. \end{aligned} \tag{2.17}$$

Using the polynomial approximation (2.17) for $g(\tau)$ in the integrals (2.5) and following the procedure adopted in Section 2.1 in determining the modified Taylor-series method (2.8), we obtain the following explicit method

$$y_{n+1} = y_n + hy'_n\,\frac{\sin\omega}{\omega} + h^2[AF_2 + BhF_3 + Ch^2F_4], \tag{2.18a}$$

$$y'_{n+1} = y'_n\cos\omega + h[AF_1 + BhF_2 + Ch^2F_3], \tag{2.18b}$$

which for $a_2 \neq a_3 \neq a_4$ can be written in the modified Runge-Kutta form (2.13) for $M = 4$ with

$$\begin{aligned} W_1 &= 2F_2 - \{2(a_3^2 - a_2^2)F_3 - 4(a_3 - a_2)F_4\}/A_4, \\ W_2 &= (2a_3^2F_3 - 4a_3F_4)/A_4, \\ W_3 &= (-2a_2^2F_3 + 4a_2F_4)/A_4, \\ W_4 &= 0, \end{aligned} \tag{2.19}$$

$$W_1^* = 2F_1 - \{2(a_3^2 - a_2^2)F_2 - 4(a_3 - a_2)F_3\}/A_4,$$

$$W_2^* = (2a_3^2 F_2 - 4a_3 F_3)/A_4,$$

$$W_3^* = (-2a_2^2 F_2 + 4a_2 F_3)/A_4,$$

$$W_4^* = 0.$$

If we make use of the classical nodes and the values of A, B, C, p as given in (2.16), we can write (2.18) in the form (2.13) for $M = 4$ with

$$\begin{aligned} W_1 &= 2F_2 - 6F_3 + 8F_4, \\ W_2 &= W_3 = 4(F_3 - 2F_4), \\ W_4 &= -2(F_3 - 4F_4), \\ W_1^* &= 2F_1 - 6F_2 + 8F_3, \\ W_2^* &= W_3^* = 4(F_2 - 2F_3), \\ W_4^* &= -2(F_2 - 4F_3). \end{aligned} \tag{2.20}$$

3. Accuracy and stability

The order q of a singlestep method is defined to be the number q for which the coefficients of the method agree with the coefficients in the Taylor-series expansion of the solution $y(t_n+h)$ up to the term h^q. We note that the explicit methods (2.13) for $M = 4$ become methods of $O(h^4)$ and $O(h^2)$ as $p \to 0$ and as $p \to \infty$ respectively. The Runge-Kutta-type methods (2.19) are of $O(h^3)$ when $p = 0$ and of $O(h)$ as $p \to \infty$.

Applying the modified singlestep method developed above to the test equation

$$y'' = -\lambda y, \quad \lambda > 0, \tag{3.1}$$

we find that when p is estimated as given in (2.15) or p is chosen as the square of the frequency of the solution of the linear homogeneous problem in (3.1), viz. $p = \lambda$,

$$\begin{aligned} \varphi_i &= f_i + py_i \\ &= -\lambda y_i + py_i = 0, \quad \text{for } i = 1, 2, 3, 4, \end{aligned}$$

and so the method becomes

$$y_{n+1} = y_n \cos\omega + hy'_n \frac{\sin\omega}{\omega},$$

$$hy'_{n+1} = -y_n \omega \sin\omega + hy'_n \cos\omega,$$

which can be written as

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix} = \begin{bmatrix} \cos\omega & \frac{\sin\omega}{\omega} \\ -\omega\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \end{bmatrix} \tag{3.2}$$

The characteristic equation of the method is

$$\xi^2 - 2\cos\omega\,\xi + 1 = 0. \tag{3.3}$$

The roots of (3.3) are complex and are of unit modulus and hence by the definition (1.2), the method is P-stable.

4. Numerical results

We use the method (2.13) for $M = 4$ and the Runge-Kutta-Treanor method (2.19), with Nyström nodes (see Henrici [3]) and Lobatto nodes to find the numerical solution of the following initial-value problems:

(i) $y'' + (100 + \frac{1}{4t^2})y = 0$

with the exact solution

$y(t) = \sqrt{t}\, J_0(10t)$ where J_0 is the Bessel function of order zero.

(ii) $x'' = -\frac{x}{r^3}, \quad y'' = -\frac{y}{r^3}$

with the exact solution $x = \cos t$, $y = \sin t$ where $r^2 = x^2 + y^2$.

The error values $E = |y(t_n) - y_n|$ are tabulated in Table I. We choose p as follows:

for problem I:

(i) as given by (2.15)

and

(ii) as $p_n = \lambda_n(t_n) = 100 + \dfrac{1}{4t_n^2}$,

and for problem II:

(i) as given by (2.15)

and

(ii) as $p_n = \lambda_n(t_n) = \dfrac{1}{r_n^3}$.

The modified methods produce superior results compared with the corresponding classical methods (*iii*) $p = 0$, for the choice (*ii*), viz. $p_n = \lambda_n(t_n)$. However, it is noticed that the choice of p given by (*i*) gives results of almost the same accuracy for problem I, whereas for problem II the choice (*ii*) gives more accurate results.

TABLE I

METHODS h		NYSTRÖM-RK4 (2.13)	LOBATTO-RK4 (2.13)	RK-NYSTRÖM-TREANOR (2.18)	RK-LOBATTO-TREANOR (2.18)
$y'' + \left(100 + \dfrac{1}{4t^2}\right) y = 0,\ y(t) = \sqrt{t}\,J_0(10t)$. Absolute error at $t = 6$:					
0.2	(i)	0.4868(−05)	0.3257(−03)	0.4868(−05)	0.3257(−03)
	(ii)	0.4174(−05)	0.3477(−04)	0.4417(−04)	0.7120(−05)
	(iii)	0.1470	0.1999	0.2372	0.2388
0.25	(i)	0.8081(−05)	0.3881(−04)	0.8081(−05)	0.3881(−04)
	(ii)	0.3308(−04)	0.1197(−03)	0.1458(−03)	0.6222(−04)
	(iii)	0.3661	0.2254	0.8465(+04)	0.2274
0.5	(i)	0.3229(−04)	0.1200(−03)	0.3229(−04)	0.1200(−03)
	(ii)	0.1816(−02)	0.2172(−03)	0.3832(−02)	0.3209(−03)
	(iii)	0.2279(+09)	0.2555(+10)	0.1155(+10)	0.2460(+10)
$\begin{cases} x'' = -x/r^3 \\ y'' = -y/r^3 \end{cases}$		$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$	Absolute error in radius at $t = 12\pi$:		
$\dfrac{\pi}{18}$	(i)	0.1095(−05)	0.1491(−03)	0.1322(−04)	0.1389(−03)
	(ii)	0.3404(−06)	0.8612(−07)	0.1076(−03)	0.1875(−04)
	(iii)	0.1324(−04)	0.1151(−04)	0.1725(−01)	0.7862(−03)
$\dfrac{\pi}{15}$	(i)	0.5971(−06)	0.3203(−03)	0.3002(−04)	0.3003(−03)
	(ii)	0.1201(−05)	0.3255(−06)	0.2643(−03)	0.4654(−04)
	(iii)	0.3258(−04)	0.2855(−04)	0.3065(−01)	0.1356(−02)
$\dfrac{\pi}{10}$	(i)	0.7567(−05)	0.5917(−02)	0.4294(−03)	0.5566(−02)
	(ii)	0.2033(−04)	0.5508(−05)	0.1907(−02)	0.3494(−03)
	(iii)	0.2361(−03)	0.2144(−03)	0.1350	0.4541(−02)

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